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SOLUTION OF A TRANSCENDENTAL EQUATION ENCOUNTERED IN DIFFRACTION PROBLEMS*

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An analytic expression is obtained for certain solutions of the transcendental equation encountered in the course of solving the diffraction problem for elastic waves on a circular cylinder and sphere, in the high frequency case. A new expression for the Hankel function is obtained in the region where the argument is large in modulo and the exponent is of the order of the argument.

Certain cases of solution to the problem of diffraction of elastic waves on a sphere or a circular cylinder, lead to the equation

$$\varphi_4 F(a) F(ma) + \varphi_3 F(a) + \varphi_2 F(ma) + \varphi_1 = 0 \tag{1}$$

$$F(z) = H_\nu^{(1)\prime}(z)/H_\nu^{(1)}(z), \quad H_\nu^{(1)\prime}(a) = \left.\frac{dH_\nu^{(1)}(z)}{dz}\right|_{z=a} \tag{2}$$

$$\varphi_1 = \lambda^2\left(1 - 2\frac{\nu^2}{a^2}\right)^2 - \varkappa^2\frac{\nu^2}{a^4}; \quad \varphi_2 = \frac{m}{a}, \quad \varphi_3 = \frac{1}{a}$$

$$\varphi_4 = -\frac{4m\nu^2}{a^2} + \frac{\varkappa^2}{a^2}; \quad 0 < m < 1$$

Here $H_\nu^{(1)}(z)$ denotes the Hankel function of the first kind, of order ν. Both the argument and the index ν are, generally speaking, complex quantities sufficiently large in modulo so that an asymptotic expansion can be used. The quantity a is variable and is an asymptotically large while ν is unknown. A particular case of (1) when $\lambda^2 = 1, \varkappa = \pm 3$, is dealt with in /1,2/. However, the misprints, omissions and coarse examples appearing in these papers prevent us from using their results uncritically. The purpose of this paper is to solve the equation (1) again and to obtain a more reliable result, and we shall make use of many examples of solution of the problem given in /1,2/.

Following /1,2/ as well as /3,4/ we shall write $H_\nu^{(1)}(z)$ in the form of a linear combination (A and B are unknown functions)

$$H_\nu^{(1)}(z) = A(\eta) H_{1/3}^{(1)}(\xi) + B(\eta) H_{-2/3}^{(1)}(\xi)$$

where $\eta = \operatorname{th}\gamma$, $\operatorname{ch}\gamma = \nu/z$, $\xi = e^{-i\pi/2}\nu\eta^3/3$. Assuming that the ratio ν/z is almost equal to unity, we choose the branch of the function $\gamma = \operatorname{arch}(\nu/z)$ which is positive when ν/z is greater than one. Noting that /3,4/

$$\xi\frac{d}{d\xi}H_{1/3}^{(1)}(\xi) = -\frac{1}{3}H_{1/3}^{(1)}(\xi) + \xi H_{-2/3}^{(1)}(\xi)$$

$$\xi\frac{d}{d\xi}H_{-2/3}^{(1)}(\xi) = -\frac{2}{3}H_{-2/3}^{(1)}(\xi) - \xi H_{1/3}^{(1)}(\xi)$$

we obtain the following relations

$$H_\nu^{(1)\cdot}(z) = \left(A^\cdot - \frac{\xi^\cdot}{3\xi}A - \xi^\cdot B\right)H_{1/3}^{(1)}(\xi) + \left(B^\cdot - \frac{2}{3}\frac{\xi^\cdot}{\xi}B + \xi^\cdot A\right)H_{-2/3}^{(1)}(\xi) \tag{3}$$

$$H_\nu^{(1)\cdot\cdot}(z) = \left\{A^{\cdot\cdot} - \frac{2}{3}\frac{\xi^\cdot}{\xi}A^\cdot + \left(\frac{4}{9}\frac{\xi^{\cdot 2}}{\xi^2} - \frac{\xi^{\cdot\cdot}}{3\xi} - \xi^{\cdot 2}\right)A - 2\xi^\cdot B^\cdot + \left(\frac{\xi^{\cdot 2}}{\xi} - \xi^{\cdot\cdot}\right)B\right\}H_{1/3}^{(1)}(\xi) + \left\{B^{\cdot\cdot} - \frac{4}{3}\frac{\xi^\cdot}{\xi}B^\cdot + \left(\frac{10}{9}\frac{\xi^{\cdot 2}}{\xi^2} - \frac{2}{3}\frac{\xi^{\cdot\cdot}}{\xi} - \xi^{\cdot 2}\right)B + 2\xi^\cdot A^\cdot - \left(\frac{\xi^{\cdot 2}}{\xi} - \xi^{\cdot\cdot}\right)A\right\}H_{-2/3}^{(1)}(\xi)$$

Here and henceforth a dot denotes differentiation with respect to z. Denoting the differentiation with respect to η by the subscript η, we substitute the relation (3) into the Bessel

*Prikl.Matem.Mekhan.,Vol.47,No.1,pp.171-174,1983 equation of order ν. Equating the coefficients of $H^{(1)}_{1/3}(\xi)$ and $H^{(1)}_{-2/3}(\xi)$ to zero, we obtain

$$A_{\eta\eta}\frac{1-\eta^2}{\eta^2}+A_\eta\frac{\eta^2-3}{\eta^3}-A\frac{\eta^2-3}{\eta^4}-A\nu^2\eta^4\frac{2-\eta^2}{1-\eta^2}= \tag{4}$$

$$2i\nu\left[B_\eta(1-\eta^2)-\frac{1}{\eta}B\right]$$

$$B_{\eta\eta}\frac{1-\eta^2}{\eta^2}+B_\eta\frac{3\eta^2-5}{\eta^3}-4B\frac{\eta^2-2}{\eta^4}-B\nu^2\eta^4\frac{2-\eta^2}{1-\eta^2}=$$

$$-2i\nu\left[A_\eta(1-\eta^2)-\frac{1}{\eta}A\right]$$

and having in mind a solution regular at the zero, we put

$$A = C(\nu)\ (\alpha_0+\alpha_1\eta^2+\alpha_2\eta^4+\ldots)\,\eta,\quad B = i\nu C(\nu)\,(\beta_3\eta^6+\beta_4\eta^8+\ldots) \tag{5}$$

where $C(\nu)$ is a multiplier independent of η and ξ.

Substituting (5) into (4) and equating the coefficients of like powers in η, we obtain an infinite system of equations for the coefficients α_l, β_l. According to /2 – 4/ we must put $\alpha_0 = 1;\ \alpha_1 = 0.1$. Then

$$\alpha_2 = 1/20,\ \alpha_3 = 1/30,\ \alpha_4 = 1/40\ldots;\ \beta_3 = 1/5,\ \beta_4 = 17/120$$

The resulting asymptotic expansions differ appreciably from those obtained in /1,2/. We also note that the series (5) converge when $|\eta| < 1$. If ρ is positive, then the clockwise passage around the zero over a straight angle, yields the relation

$$H^{(1)}_{1/3}(\rho e^{-i\pi}) = \frac{\sqrt{3}-i}{2}\,[\sqrt{3}J_{1/3}(\rho)-Y_{1/3}(\rho)] \tag{6}$$

where $J_{1/3}(\rho)$ and $Y_{1/3}(\rho)$ are the principle values of the Bessel function of first and second kind respectively. Let $\rho_k - ke$ denote in increasing order the value of the root of the expression appearing within the square brackets in (6). Then /3/ $H^{(1)}_{1/3}(\rho_k e^{-i\pi}) = 0$. Denoting $H^{(1)}_{-2/3}(\rho_k e^{-i\pi})$ by $H_k{}^*$, we obtain the following expansions of the functions $H^{(1)}_{1/3}(\xi)$ and $H^{(1)}_{-2/3}(\xi)$ into the Taylor series in powers of the difference $(\xi - \rho_k e^{-i\pi})$

$$H^{(1)}_{1/3}(\xi) = \left[(\xi-\rho_k e^{-i\pi}) - \frac{1}{2}(\rho_k e^{-i\pi})^{-1}(\xi-\rho_k e^{-i\pi})^2+\ldots\right]H_k{}^* \tag{7}$$

$$H^{(1)}_{-2/3}(\xi) = \left[1-\frac{2}{3}(\rho_k e^{-i\pi})^{-1}(\xi-\rho_k e^{-i\pi})+\frac{1}{2\rho_k{}^2}\left(\frac{10}{9}-\rho_k{}^2\right)\times\right.$$

$$\left.(\xi-\rho_k e^{-i\pi})^2+\ldots\right]H_k{}^*$$

Let us introduce the variable ε_k dependent on the argument a

$$\varepsilon_k = \left(\frac{3\rho_k}{a}\right)^{1/3} e^{-i5\pi/6} = a^{-1/3}\,(e^{-i\pi/2}3\rho_k)^{1/3} \tag{8}$$

where $(e^{-i\pi/2})^{1/3} = e^{-i5\pi/6}$. We write η in the form of an asymptotic expansion (series) in powers of the small parameter $\varepsilon_k : \eta = e_0\varepsilon_k\,(1+e_1\varepsilon_k+e_2\varepsilon_k{}^2+\ldots)$. We can put $e_0 = 1-\varepsilon_k{}^3/6$ with the accuracy of up to but not including the terms of the second order of smallness. Then $\xi - \rho_k e^{-i\pi} = 3\rho_k e^{-i\pi}\varepsilon_k\,[e_1 + (e_1{}^2+e_2)\,\varepsilon_k+\ldots]$, which on substitution into (7) yields

$$H^{(1)}_{1/3}(\xi) = 3H_k{}^*\rho_k e^{-i\pi}\varepsilon_k\left[e_1+\left(e_2-\frac{1}{2}e_1{}^2\right)\varepsilon_k+\ldots\right]$$

$$H^{(1)}_{-2/3}(\xi) = H_k{}^*\left\{1-2e_1\varepsilon_k+\left[\left(3+\frac{9}{2}\rho_k{}^2\right)e_1{}^2-2e_2\right]\varepsilon_k{}^2+\ldots\right\}$$

Using the resulting asymptotic expansion we obtain

$$F_\nu(z) = \frac{H^{(1)'}_\nu(z)}{H^{(1)}_\nu(z)} = \frac{e^{-i\pi/2}\,(1-e_1\varepsilon_k+\ldots)}{3\rho_k\left[e_1+\left(e_2-\frac{1}{2}e_1{}^2-\frac{1}{5}\right)\varepsilon_k+\ldots\right]} \tag{9}$$

At large values of η the Debye expansions of the Hankel function /3,4/ are more convenient. If e.g. $|z| \ll \nu$ and z in real, then

$$H^{(1)}_\nu(z) = -i\sqrt{\frac{2}{\pi}}\,(\nu^2-z^2)^{-1/4}\exp\left\{\nu\ln\frac{\nu+\sqrt{\nu^2-z^2}}{z}-\sqrt{\nu^2-z^2}\right\} \tag{10}$$

$$F_\nu(ma) = -\frac{\sqrt{1-m^2}}{m}\,(1+\ldots)$$

where the repeated dots denote terms of order η^2 and higher. Setting in (9) $z = a$ and substituting (9) and (10) into (1), we obtain

$$4e^{-\iota\pi/2}\frac{\sqrt{1-m^2}\,(1-e_1\varepsilon_k+\ldots)}{3\rho_k\left[e_1+\left(e_2-\frac{1}{2}e_1^2-\frac{1}{5}\right)\varepsilon_k+\ldots\right]}-\lambda^2=0$$

which, by comparing the coefficients of like powers of ε_k yields

$$e_1=-\frac{4\iota\sqrt{1-m^2}}{3\rho_k\lambda^2},\quad e_2=-\frac{1}{2}e_1^2+\frac{1}{5},\ldots$$

We further have

$$\nu_k=a\left(1+\frac{1}{2}\eta_k^2+\frac{1}{8}\eta_k^4+\ldots\right)= \tag{11}$$
$$a\left(1+\frac{1}{2}\varepsilon_k^2-\frac{4\iota\sqrt{1-m^2}}{3\rho_k\lambda^2}\varepsilon_k^3-\frac{1}{120}\varepsilon_k^4+\ldots\right)$$
$$\rho_1=2.38;\quad \rho_2=5.51;\quad \rho_3=8.65;\quad \rho_4=11.8;\quad \rho_5=14.9\ldots$$

where ε_k are given by (8) and ρ_k are defined in /2,3/. Thus we have obtained the roots of the first series. The principal part of their asymptotic expansion is equal to a. In /5/ it was shown that roots of this type must lie in the first quadrant of the complex ν-plane. It is for this reason that out of three possible values of the argument of the expression $(e^{-\iota\pi/2})^{1/3}$ in (8) we choose one which ensures that the above condition is fulfilled.

Let us now compute the roots of the second series in which the principle part of the asymptotic expansion is equal to ma. In this case $a>\nu\sim ma$ and the Hankel function $H_\nu^{(1)}(a)$ written in the form of an asymptotic Debye expansion has the logarithmic derivative $F_\nu(a)=\iota\sqrt{1-m^2}\,(1+\ldots)$. We introduce the notation $\operatorname{th}\gamma=\eta=\varepsilon_k(1-{}^1/_6\,\varepsilon_k^2)\ (1+e_1\varepsilon_k+\ldots)$, $\nu=ma\operatorname{ch}\gamma$. The remaining notation is that used to computing the roots of the first series. We note that now $\varphi_4\approx-4m^3$; $\varphi_1\approx\lambda^2(1-2m^2)^2$. Substituting the last relations into the initial equation just as we did in computing the roots of the first series, we obtain

$$\nu_k=ma\left(1+\frac{1}{2}\varepsilon_k^2+\frac{4m^3\sqrt{1-m^2}}{3\rho_k(2m^2-1)^2\lambda^2}\varepsilon_k^3-\frac{1}{120}\varepsilon_k^4+\ldots\right) \tag{12}$$

The expressions (11) and (12) obtained enable us to determine the contribution made to the diffraction field by the transverse and longitudinal type waves /5/ respectively, the waves sliding along the reflecting surface.

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